# Differentiation:-

* The concept of differentiation is a basic notion in Calculus.
* Derivatives are used to solve a wide range of problems involving tangents and rates of change.
* The motivation is based on two different types of problems:
  - (i) The physical problem of finding the instantaneous velocity of a moving particle, and
  - (ii) The geometric problem of finding the tangent line to a curve at a given point.

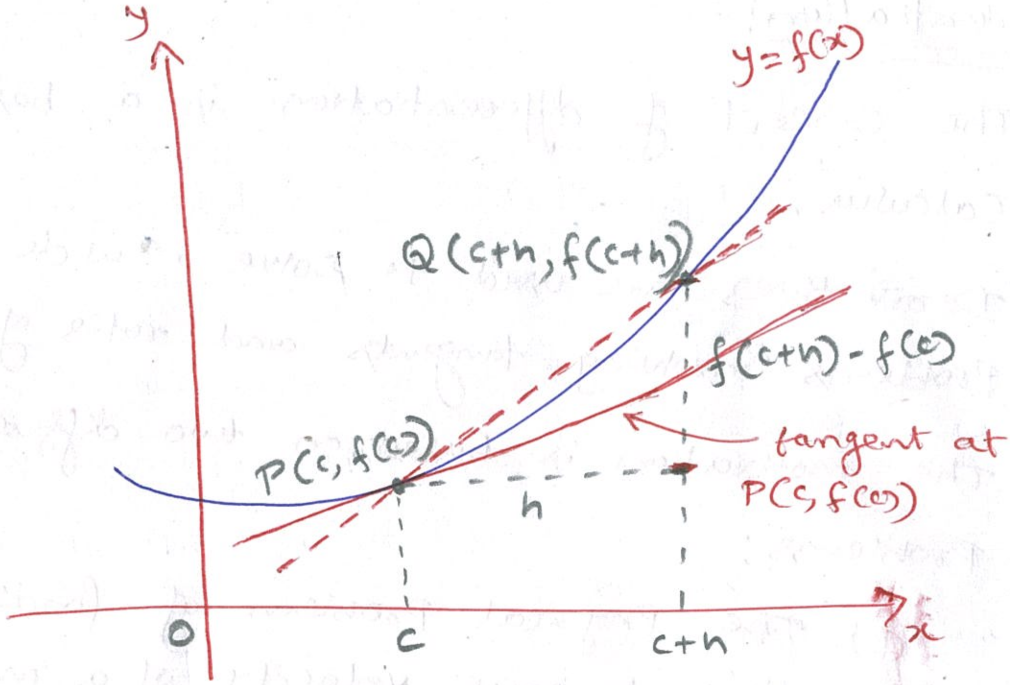
**Def:-** Let $D \subseteq \mathbb{R}$ and $c \in D$ be an interior point of D, which means $(c-\epsilon, c+\epsilon) \subseteq D$ for some $\epsilon > 0$. [Some times, instead of taking interior pts., one can define f on an open interval D.]

Let $f: D \rightarrow \mathbb{R}$ be a function. We say that f is "differentiable" at c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

In this case, the value of this limit is denoted by $f'(c)$ and called the "derivative" of f at c.




The slope of the tangent line at P is

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \text{The derivative of } f \text{ at a pt. } c.$$

Intutively, $f$ is differentiable at $c$ means the curve $y = f(x)$ has a unique non-vertical tangent at $(c, f(c))$.

To be specific, if $f : D \to \mathbb{R}$ is differentiable at a pt. $c$ of $D$, then we define the "tangent" to the curve $y = f(x)$ at the point $(c, f(c))$ to be the line given by $\boxed{f(x) - f(c) = (x - c) f'(c)}$

ie, $y - f(c) = (x - c) f'(c)$.

## Examples :-

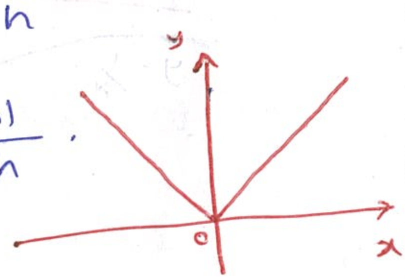
1. If $f: \mathbb{R} \to \mathbb{R}$ is a constant function, then $f$ is differentiable and $f'(c) = 0 \ \forall c \in \mathbb{R}$.

2. If $f: \mathbb{R} \to \mathbb{R}$ is the identity function, $f(x) = x$ $\forall x \in \mathbb{R}$, then $f$ is differentiable and $f'(c) = 1 \ \forall c \in \mathbb{R}$.

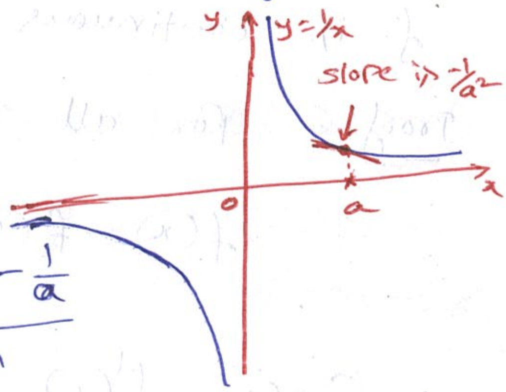
3. If $f: \mathbb{R} \to \mathbb{R}$ is the absolute value function, given by $f(x) = |x| \ \forall x \in \mathbb{R}$, then $f$ is <u>not</u> differentiable at $0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h}$$

and $\displaystyle\lim_{h \to 0^+} \frac{|h|}{h} = +1 \neq -1 = \lim_{h \to 0^-} \frac{|h|}{h}$.




4. Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What happens to the tangent to the curve at the point $(a, 1/a)$ as 'a' changes?




Sol:- Here $f(x) = 1/x$.

The slope at $(a, 1/a)$ is

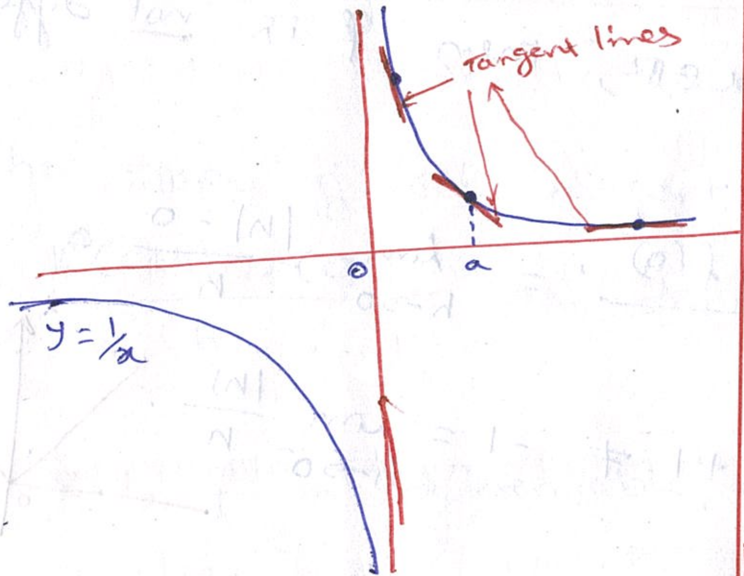
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{a - a - h}{a(a+h) \cdot h} = \lim_{h \to 0} \frac{-h}{ha(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)}$$

$$= -\frac{1}{a^2}.$$

Observe that the slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep.

Similarly as $a \to 0^-$, the slope approaches $-\infty$. But as $a$ moves away from the origin in either direction, the slope approaches $0$ and the tangent levels off to become horizontal.




**Observation!**

The following are all interpretations for $f'(c)$:

(i). The slope of the graph of $y = f(x)$ at $x = c$.

(ii). The slope of the tangent to the curve $y = f(x)$ at $x = c$.

(iii). The rate of change of $f(x)$ w.r.t $x$ at $x = c$.

Here
$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

**Result:-**

1. If $f : D \to \mathbb{R}$ has a derivative at $c \in D$, then $f$ is continuous at $c$.

**Proof:-** For all $x \in D$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{(x-c)}\right)(x-c).$$

Since $f'(c)$ exists, applying limit of a Product, we have

$$\lim_{x \to c} \left(f(x) - f(c)\right) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x-c}\right) \cdot \left(\lim_{x \to c} (x-c)\right)$$
$$= f'(c) \cdot 0 = 0$$

i.e. $\lim_{x \to c} f(x) = f(c)$

$\therefore$ $f$ is continuous at $c$.

2. $\boxed{\text{Carathéodory's theorem}}$

Let $f$ be defined on an interval $I$ containing the point $c$. Then $f$ is differentiable at $c$ "if and only if" $\exists$ a function $\phi : I \to \mathbb{R}$ such that

(i) $f(x) - f(c) = \phi(x)(x-c)$ for $x \in I$.

(ii) $\phi$ is continuous at $c$.

In this case, we have $\phi(c) = f'(c)$.

Proof:- $\Rightarrow$ If $f'(c)$ exists, we can define $\phi$ by

$$\phi(x) = \begin{cases} \dfrac{f(x) - f(c)}{x-c} & \text{for } x \neq c,\ x \in I \\ f'(c) & \text{for } x = c. \end{cases}$$

The continuity of $\phi$ follows from the fact that $\lim_{x \to c} \phi(x) = f'(c)$.

If $x = c$, then $f(x) - f(c) = 0 = \phi(x)(x-c)$, while if $x \neq c$, then $\phi(x)(x-c) = f(x) - f(c)$ $\forall$ other $x \in I$. (by the way we defined $\phi$).

$\Leftarrow$ Now assume a cont fn, $\phi$ at $c$ that satisfies $f(x) - f(c) = \phi(x)(x-c)$, for $x \in I$.

Now for $x - c \neq 0$, $\phi(x) = \dfrac{f(x) - f(c)}{x - c}$.

The continuity of $\alpha$ at $c$ gives that

$$\alpha(c) = \lim_{x \to c} \alpha(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists}$$

$\therefore$ $f$ is differentiable at $c$ and $f'(c) = \alpha(c)$.

*

**Note:-**

1. Carathéodory's theorem gives us a very nice method for proving the "chain rule".

2. Carathéodory's theorem will also be used to derive the formula for "differentiating inverse functions".

3. If a function $\alpha$ satisfying (i) & (ii) as in Carathéodory's theorem exists, then it is unique and we call it the "increment function" associated to $f$ and $c$.

   Hence, derivative of $f$ at $c$ = the value of the increment function at $c$.

**Basic Properties:**

If $f$ and $g$ are differentiable at '$c$',

then (i) $f+g$ is differentiable at '$c$' and

$$(f+g)'(c) = f'(c) + g'(c).$$

ii, $Kf$ is differentiable at $c$ and

$$(Kf)'(c) = K f'(c) \text{ for any } K \in \mathbb{R}.$$

iii, $f \cdot g$ is differentiable at $c$ and

$$(fg)'(c) = f'(c) g(c) + f(c) g'(c).$$

iv, If $g(c) \neq 0 \;\forall\; x \in$ Domain of $g$, and the fn. $\frac{1}{g}$ is differentiable at $c$, then

$$\left(\frac{1}{g}\right)'(c) = \frac{-g'(c)}{[g(c)]^2}.$$

In general, $$\left(\frac{f}{g}\right)'(c) = \frac{g(c) f'(c) - f(c) g'(c)}{[g(c)]^2}.$$

v, If $f(c) > 0$ ~~$\forall x$~~ then $\exists\, \delta > 0 \ni f(x) > 0 \;\forall x$, ~~and~~ ~~then~~ and the function $f^{1/K}$ is differentiable at $c$, ~~then~~ and

$$\left(f^{1/K}\right)'(c) = \frac{1}{K} f(c)^{\left(\frac{1}{K}-1\right)} f'(c).$$

## The Chain Rule:

Let $I, J$ be intervals in $\mathbb{R}$, let $g: I \to \mathbb{R}$ and $f: J \to \mathbb{R}$ be functions $\ni f(J) \subseteq I$, and let $c \in J$. If $f$ is differentiable at $c$ and if $g$ is differentiable at $f(c)$, then the composition function $g \circ f$ is differentiable at $c$ and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

## Proof of Chain Rule:-

Since $f'(c)$ exists, by Carathéodory's theorem, there exists incremental fn. $\varphi$ on $J$ such that $\varphi$ is continuous at $c$ and

$$f(x) - f(c) = \varphi(x)(x-c) \text{ for } x \in J,$$

and where $\varphi(c) = f'(c)$.

Similarly, since $g'(f(c))$ exists, $\exists$ incremental fn. $\psi$ defined on $I \ni \psi$ is continuous at $f(c)(=d)$ and

$$g(y) - g(d) = \psi(y)(y-d) \text{ for } y \in I,$$

where $\psi(d) = g'(d)$.

Substituting $y = f(x)$ and $d = f(c)$, we have

$$g(f(x)) - g(f(c)) = \psi(f(x))(f(x) - f(c))$$

$$= \psi(f(x))\,\varphi(x)(x-c).$$

$$\therefore (g \circ f)(x) - (g \circ f)(c) = [(\psi \circ f)(x) \cdot \varphi(x)](x-c).$$

$\forall x \in J \ni f(x) \in I.$

$\therefore (\psi \circ f) \cdot \varphi$ is an incremental function for $g \circ f$ at $c$. (Since the fn. $(\psi \circ f) \cdot \varphi$ is continuous at $c$).

$$\boxed{\therefore (g \circ f)'(c) = \psi(f(c)) \cdot \varphi(c) = g'(f(c)) \cdot f'(c)}$$

Note:-

1. Statement of Chain Rule in Leibnitz's notion is,

if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $dy/du$ is evaluated at $u = g(x)$.

2. If $f$ is a differentiable function of $u$ and if $u$ is a differentiable function of $x$, then substituting $y = f(u)$ into the chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u) \cdot \frac{du}{dx}.$$

If $n$ is any real number and $f$ is a Power function, $f(u) = u^n$, the Power rule tells us that

$f'(u) = nu^{n-1}$.

If $u$ is a differentiable function of $x$, then we can use the chain Rule to extend this to the "Power Chain Rule":

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \quad \left( \because \frac{d}{du}(u^n) = nu^{n-1} \right)$$

Examples:-

1. Suppose that $f: I \longrightarrow \mathbb{R}$ is differentiable on $I$ such that $f(x) \neq 0$ and $f'(x) \neq 0$ for $x \in I$. Using Chain Rule, show that $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$ for $x \in I$.

Sol:- Consider $g(y) = 1/y$ for $y \neq 0$, then using quotient rule, it is easy to see that

$$g'(y) = -1/y^2 \text{ for } y \in \mathbb{R},\ y \neq 0.$$

Now
$$\left(\frac{1}{f}\right)'(x) = [g(f(x))]' = (g \circ f)'(x)$$
$$= g'(f(x)) \cdot f'(x) \quad \text{(by Chain Rule)}$$
$$= -\frac{f'(x)}{[f(x)]^2} \quad \text{for } x \in I.$$

2. An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2+1)$. Find the velocity of the object as a function of $t$.

Sol:- Velocity $= \frac{dx}{dt}$.

Here, $x$ is a composite function. ie,

$$x = \cos(u) \text{ and } u = t^2+1.$$

We have $\frac{dx}{du} = -\sin(u)$

& $\frac{du}{dt} = 2t$.

∴ By the chain Rule,

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$

$$= -\sin(u) \cdot 2t$$

$$= -\sin(t^2+1) \cdot 2t$$

$$= -2t \sin(t^2+1).$$

## Implicit Differentiation

Def:- (Implicit function)

If a function in which the dependent variable is expressed solely in terms of the independent variable $x$, namely, $y = f(x)$, then such a function is said to be in "explicit" form otherwise, we say that the function is in "implicit" form.

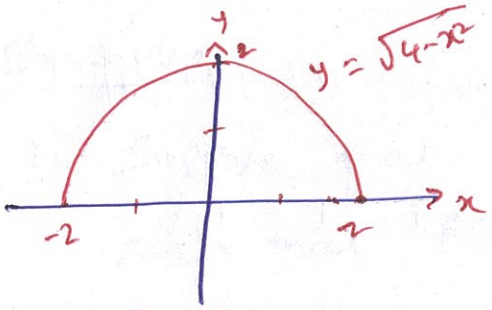
Example:-

1. $y = \frac{1}{2}x^3 - 1$ is in explicit form.

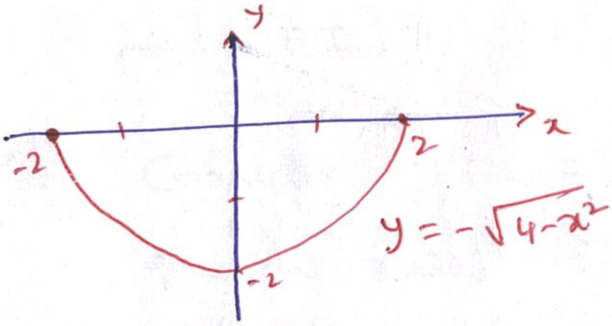
   The same function, if we express as

   $2y - x^3 + 2 = 0$ is said to be in "implicit" form.

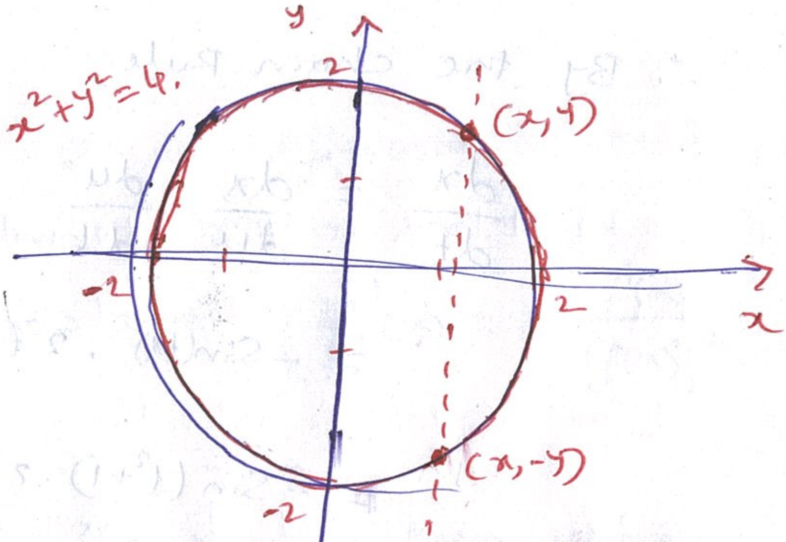
2. The equation $x^2 + y^2 = 4$ defines the two functions $f(x) = \sqrt{4-x^2}$ and $g(x) = -\sqrt{4-x^2}$




Function




Function.




Not a function.

* $x^2+y^2=4$ defines two functions $f(x) = \sqrt{4-x^2}$ and $g(x) = -\sqrt{4-x^2}$ implicitly.

## Implicit differentiation:

The process of determining the derivative for implicit functions is known as implicit differentiation. This process consists of differentiating both sides of an equation w.r.t $x$, using the rules of differentiation and then solving for $dy/dx$.

To be specific, we follow the steps as below:

(i) Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.

(ii) Collect the terms with $dy/dx$ on one side of the equation and solve for $dy/dx$.

**Example:-**

1. Find $\frac{dy}{dx}$ if $x^2+y^2=4$.

**Sol:-** Differentiate both sides of the equation $x^2+y^2=4$ with respect to $x$.

$$\therefore \quad \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(4)$$

$$\Rightarrow \quad 2x + 2y \cdot \frac{dy}{dx} = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

## Derivatives of Inverse functions:

**Recall:** A function $f: I \to \mathbb{R}$ has an inverse function

$\Longleftrightarrow$ $f$ is injective (one-one).

(Here $I$ is an interval.)

**Note:-** A strictly monotone function is injective and so has an inverse.

## Continuous inverse Theorem:

Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be strictly monotone and continuous on $I$. Then the function $g$, inverse of $f$, is strictly monotone and continuous on $J = f(I)$.

## Differentiable Inverse theorem:-

Let $I$ be an interval in $\mathbb{R}$ and let $f: I \to \mathbb{R}$ be strictly monotone and continuous on $I$. Let $f^{-1}: f(I) \to \mathbb{R}$ be the strictly monotone and continuous function inverse of $f$.

If $f$ is differentiable at $c \in I$ and $f'(c) \neq 0$, then $f^{-1}$ is differentiable at $d = f(c)$ and

$$\cancel{(f^{-1}(d))'} \quad (f^{-1})'(d) = \frac{1}{f'(c)} = \frac{1}{f'(f^{-1}(d))}.$$

**Proof:-** (Brief outline proof)
We know that $f(f^{-1}(x)) = x$ for every $x$ in the domain of $f^{-1}$.

By implicit differentiation and Chain Rule,

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x.$$

$$\Rightarrow f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

$$\Rightarrow f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

* A more theoretical proof can be established using Caratheodory's theorem.

Example:-

Consider the function $f(x) = x^2+1$ for $x \geq 0$. Find $f^{-1}(x)$ and the derivative of $f^{-1}(x)$.

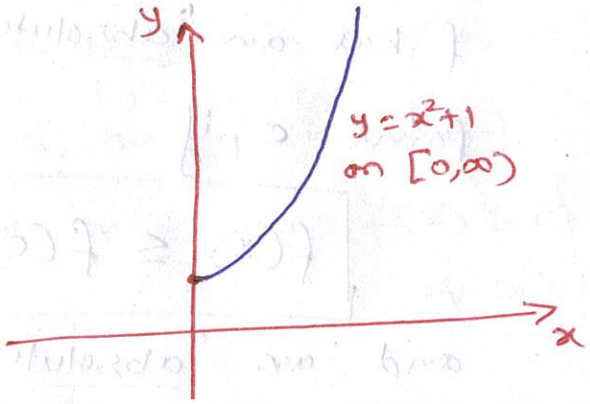
Sol:-

Solving $y = x^2+1$ for $x$,

we have

$$x^2 = y-1$$

$$\Rightarrow x = \pm\sqrt{y-1}$$

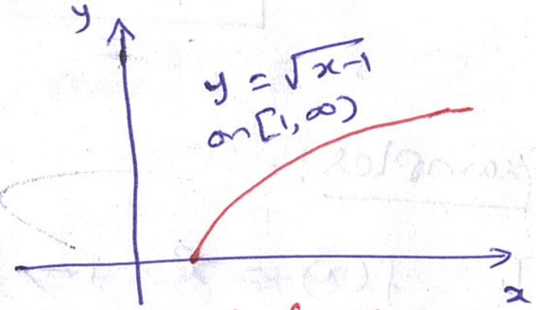



One-to-one function, Fig(i)

By relabeling variables,

we have

$$y = \pm\sqrt{x-1}.$$




Inverse of function given in Fig(i).

i.e., $f^{-1}(x) = \pm\sqrt{x-1}$ as the inverse of $f$.

Since the domain and range of $f^{-1}$ are $[1,\infty)$ and $[0,\infty)$ respectively,

we have $f^{-1}(x) = \sqrt{x-1}$ as the inverse of $f$.

Now as $f'(x) = 2x$, we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt{x-1})} = \frac{1}{2\sqrt{x-1}}.$$

# Extreme values of functions

**Def:-** Let $f$ be a function with domain $D$. Then $f$ has an "absolute maximum" value on $D$ at a point $c$ if

$$\boxed{f(x) \le f(c) \quad \forall\ x \in D}$$

and an "absolute minimum" value on $D$ at $c$ if

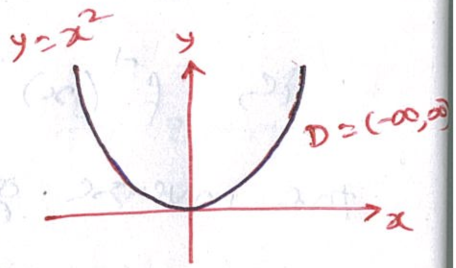
$$\boxed{f(x) \ge f(c) \quad \forall\ x \in D.}$$

**Examples:-**

1. $f(x) = x^2$ in $(-\infty, \infty)$

   No absolute maximum

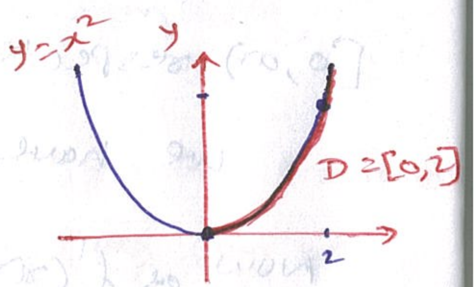
   Absolute minimum of $0$ at $x=0$.




2. $f(x) = x^2$ on $[0, 2]$

   Absolute maximum of $4$ at $x=2$

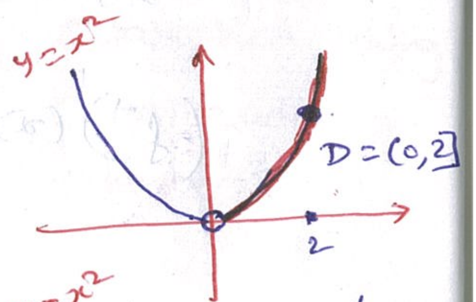
   Absolute minimum of $0$ at $x=0$.




3. $f(x) = x^2$ on $(0, 2]$

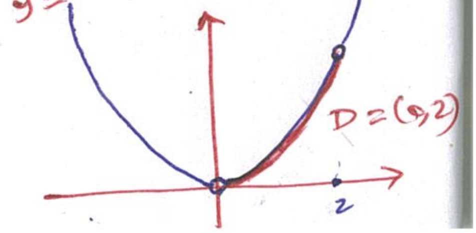
   Absolute maximum of $4$ at $x=2$

   No absolute minimum.




4. $f(x) = x^2$ on $(0, 2)$

   No absolute extrema.



## Extreme value theorem:

If $f$ is continuous on a closed interval $[a,b]$ then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a,b]$.

i.e, there are numbers $x_1$ and $x_2$ in $[a,b]$

$$\ni \; f(x_1) = m, \quad f(x_2) = M \quad \text{and} \quad m \le f(x) \le M \;\; \forall x \in [a,b].$$

**Def:-** Let $D \subseteq \mathbb{R}$ and $c \in D$. We say that a function $f: D \to \mathbb{R}$ has

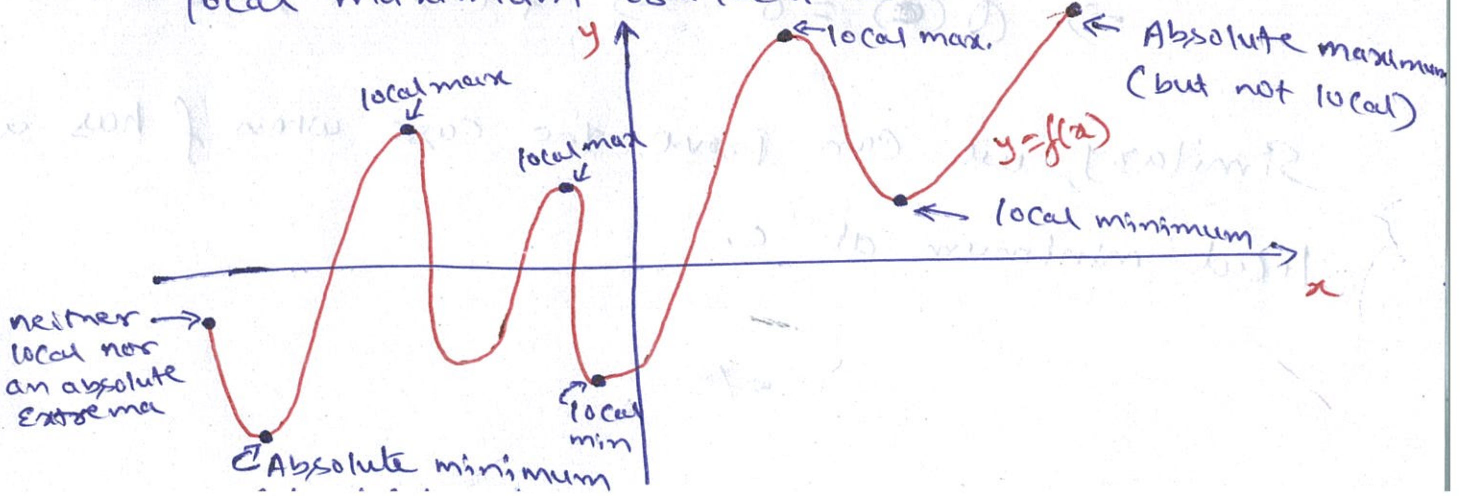
- **local minimum** at $c$ if there is $\delta > 0 \ni$ $(c-\delta, c+\delta) \subseteq D$ and

$$f(x) \ge f(c) \quad \forall x \in (c-\delta, c+\delta).$$

- **local maximum** at $c$ if there is $\delta > 0 \ni$ $(c-\delta, c+\delta) \subseteq D$ and

$$f(x) \le f(c) \quad \forall x \in (c-\delta, c+\delta).$$

- **local Extremum** at $c$ if ~~there~~ it has a local maximum or a local minimum at $c$.



## The First Derivative theorem for local Extreme values:

Let $D \subseteq \mathbb{R}$ and $c$ be an interior point of $D$. If $f: D \to \mathbb{R}$ has a local maximum or local minimum value at $c$, and if $f$ is differentiable at $c$, then

$$f'(c) = 0.$$

Proof:- By Caratheodory's theorem, $\exists$ $\varphi: D \to \mathbb{R}$

$\ni$ $f(x) - f(c) = (x-c)\varphi(x)$ $\forall$ $x \in D$, and $\varphi$ is continuous at $c$.

If $f$ has a local maximum at $c$, then $\exists$ $\delta > 0$

$\ni$ $(c-\delta, c+\delta) \subseteq D$ and

$$f(x) \leq f(c) \quad \forall\ x \in (c-\delta, c+\delta).$$

$$\Rightarrow \varphi(x) \geq 0 \quad \forall\ x \in (c-\delta, c) \ \&$$

$$\varphi(x) \leq 0 \quad \forall\ x \in (c, c+\delta)$$

$$\Rightarrow \varphi(c) = 0 \Rightarrow f'(c) = 0, \text{ as desired!}$$

Similarly, we can prove the case when $f$ has a local minimum at $c$.

Note:-

1. The first derivative theorem for local Extreme values says that a functions first derivative is always zero at an interior point where the function has a local Extreme value and the derivative is defined.

2. The only places where a function f can possibly have an Extreme value (local or global) are
   i, interior pts where $f' = 0$
   ii, interior pts where $f'$ is undefined
   iii, End points of the domain of f.

Defn:- (Critical points)

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. An interior point c of D is called a "critical point" of f if

Either f is differentiable at c and $f'(c) = 0$

or f is not differentiable at c.

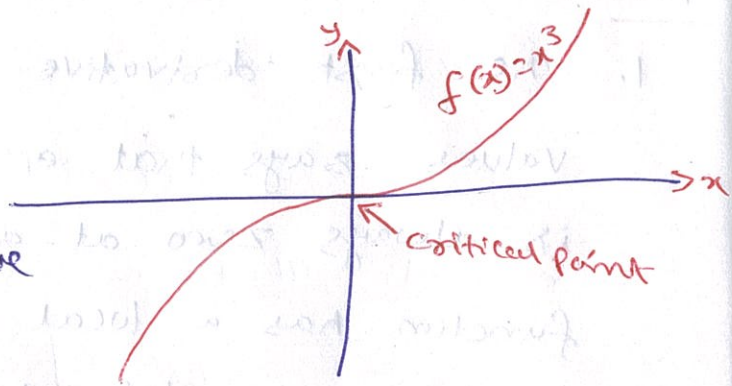
Consequences:

1. Local Extrema can only occur at critical points.

2. A function may have a critical point at $x = c$ without having a local Extreme value there.

ie., not every critical point is a local maximum

Consider $f(x) = x^3$, which has a critical point at $x=0$.



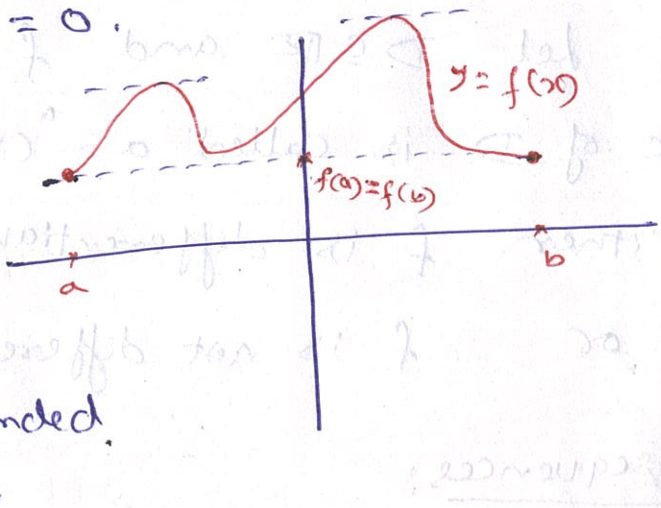

The derivative

$f'(x) = 3x^2$, is positive on both sides of $x=0$,

so $f$ increases on both sides of $x=0$, and there is neither a local maximum nor a local minimum at $x=0$.

Rolle's theorem:-

Suppose $f:[a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$ and differentiable on $(a,b)$, and that $f(a)=f(b)=0$. Then $\exists$ at least one point $c \in (a,b) \ni f'(c)=0$.




Proof:-

Since $f:[a,b] \to \mathbb{R}$ is continuous, it is bounded and attains its bounds.

Let $M = \sup\{f(x) \mid x \in [a,b]\}$ &

$m = \inf\{f(x) \mid x \in [a,b]\}$.

If both $M$ and $m$ were attained at the end points $a, b$, then $M = m$ $(\because f(a) = f(b))$.

Hence $f$ is a constant function and so

$f'(x) = 0 \;\forall\; x \in [a,b]$. In this case, any $c \in (a,b)$

will satisfy that $f'(c) = 0$.

If $M$ or $m$ were attained at an interior point,

say $c \in (a,b)$, then $c$ is a local extremum for $f$

and hence by "first derivative th for local extreme

values", ~~and hence~~ $f'(c) = 0$.

This proves the theorem.

Examples:- (Applications of Rolle's theorem)

1. Prove that there is no value of $k$ such that the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0,1]$.

Sol:- Let us suppose on the contrary that there is a real number $k'$ $\ni$ $x^3 - 3x + k' = 0$ has two distinct roots $\alpha_1$ and $\alpha_2$ in $[0,1]$, where $\alpha_1 < \alpha_2$ (W.O.L.O.G)

Now $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in [0,1] \Rightarrow 0 \le \alpha_1 < 1$ and $0 < \alpha_2 \le 1$.

Since $\alpha_1$ and $\alpha_2$ are the roots of the eq.

$x^3 - 3x + k' = 0$, we have $\left.\begin{array}{l}\alpha_1^3 - 3\alpha_1 + k' = 0 \\ \& \;\; \alpha_2^3 - 3\alpha_2 + k' = 0.\end{array}\right\}$ ---(i)

Now consider the function $f(x)$ defined in $[\alpha_1, \alpha_2]$

as follows: $f(x) = x^3 - 3x + k'$ for $x \in (\alpha_1, \alpha_2)$.

Since $f(x)$ is a polynomial in $x$ of degree 3, it is continuous on $[\alpha_1, \alpha_2]$ and is derivable in $(\alpha_1, \alpha_2)$ with $f(\alpha_1) = f(\alpha_2) = 0$. (by (i)).

Hence, all the conditions of Rolle's theorem are satisfied by $f(x)$ in $[\alpha_1, \alpha_2]$.

$\therefore \exists\, c \in (\alpha_1, \alpha_2) \ni f'(c) = 0$.

$$\Rightarrow 3c^2 - 3 = 0 \Rightarrow c = \pm 1 \notin (0,1).$$

$\therefore c \in (\alpha_1, \alpha_2)$ contradicts our assumption that $0 \le \alpha_1 < 1$ and $0 < \alpha_2 \le 1$.

$\therefore \exists$ no real number $K$ for which the given eqn. $x^3 - 3x + K = 0$ has two distinct roots in $[0,1]$.

2. Show that $x^3 + 11x + 17 = 0$ has a unique real root.

Sol:- Suppose $f(x) = x^3 + 11x + 17$

Observe that $f(-2) = -8 + (-22) + 17 = -13 < 0$

& $f(0) = 17 > 0$.

$\therefore$ we have $f(-2) < 0 < f(0)$ and $f$ is con (as the given fn. is a polynomial fn. of degree 3).

Hence by I.V.P $\exists\, c \in \mathbb{R}$ between $(-2, 0) \ni$ $f(c) = 0 \Rightarrow c$ is a root of eq. $x^3 + 11x + 17 = 0$

... at least one root in $(-2, 0)$

Suppose $f$ has more than one real root, say $a, b \in \mathbb{R} \ni a < b$. $\therefore f(a) = 0 = f(b)$.

So, by Rolle's Th $\exists\, c \in (a,b) \ni f'(c) = 0$.

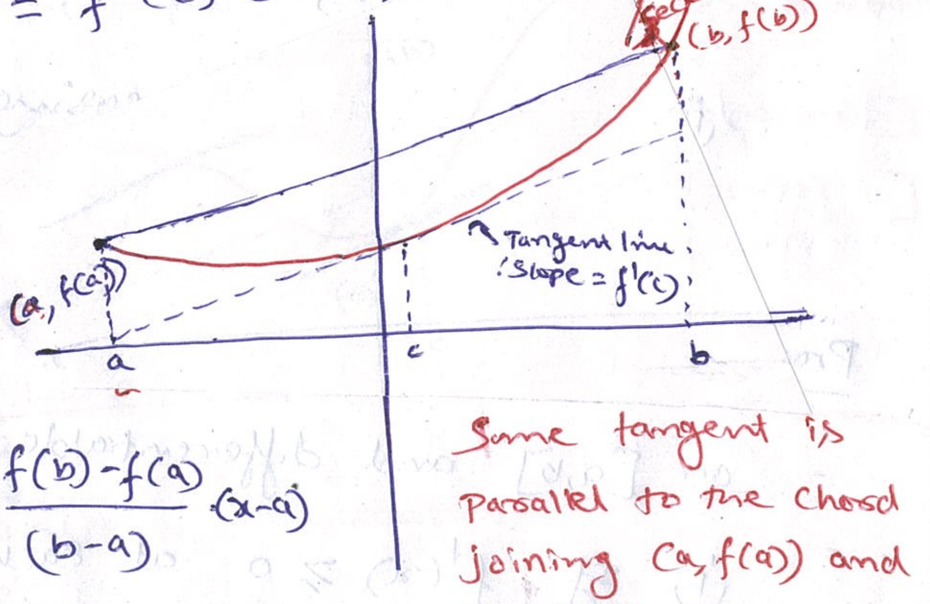
But $f'(c) = 3c^2 + 11 > 0 \quad \forall\, x \in \mathbb{R}$.

$\therefore f$ has a unique real root.

## Mean Value Theorem:- (MVT)

Suppose $f : [a,b] \to \mathbb{R}$ be a continuous fn. ~~Then~~ on $[a,b]$ and differentiable on $(a,b)$. Then $\exists$ at least one pt. $c \in (a,b) \ni$

$$f(b) - f(a) = f'(c)(b-a)$$




Same tangent is parallel to the chord joining $(a, f(a))$ and $(b, f(b))$.

Proof:-

Consider the fn. $Q$ defined on $[a,b]$ by

$$Q(x) = f(x) - f(a) - \frac{f(b)-f(a)}{(b-a)}(x-a)$$

$Q$ is continuous on $[a,b]$, differentiable on $(a,b)$ and $Q(a) = Q(b) = 0$. Hence by Rolle's Th $\exists\, c \in (a,b) \ni 0 = Q'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$.

Hence, $f(b) - f(a) = (b-a) \cdot f'(c)$.

## Consequence of MVT:

1. If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ & differentiable on $(a,b)$, and that $f'(x) = 0$ for $x \in (a,b)$, then $f$ is constant on $[a,b]$.

Hint: By M.V.T $f(x) - f(a) = f'(c)(x-a) = 0 \Rightarrow f(x) = f(a) \ \forall x$

## Monotonic functions, convexity and concavity:

Let $I$ be an interval and $f : I \to \mathbb{R}$ be any function. We say that $f$ is

i, monotonically increasing fn. on $I$ if

$$x_1, x_2 \in I, \ x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2).$$

ii, monotonically decreasing fn. on $I$ if

$$x_1, x_2 \in I, \ x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2).$$

**Prop:-** Let $I$ be an interval $[a,b]$, $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then

i, If $f'(x) \geq 0$ at each pt. on $(a,b)$, then $f$ is monotonically increasing on $[a,b]$.

ii, If $f'(x) \leq 0$ at each pt. on $(a,b)$, then $f$ is monotonically decreasing on $[a,b]$

Proof:- Hint: Use MVT

Example:-

Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

$$f'(x) = 3x^2 - 18x - 48$$

To find where $f' > 0$ or $f' < 0$, we first find where $f' = 0$ ie, $3x^2 - 18x - 48 = 0$

$$\Rightarrow 3(x+2)(x-8) = 0$$

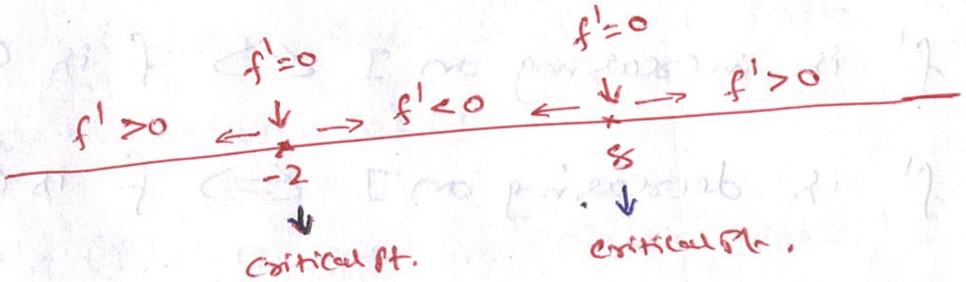
$\Rightarrow$ At $x = -2$ or $x = 8$, $f' = 0$.

$\because$ $f'$ is continuous $f'$ cannot change sign on any of the intervals $x < -2$, or $-2 < x < 8$ or $8 < x$.

Observe that $f' > 0$ for $x < -2$ [check by taking a pt. $x < -2$ & substituting in $f'$]

$f' < 0$ for $-2 < x < 8$ [ " $(-2<x<8)$ ]

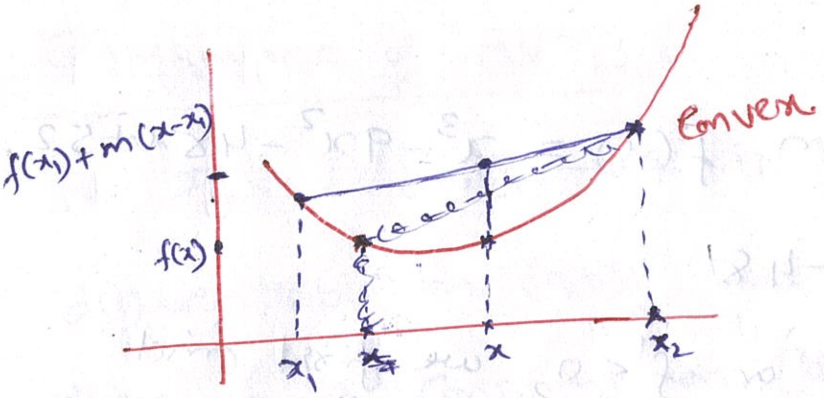
and $f' > 0$ for $x > 8$. [ " $(x>8)$ ]

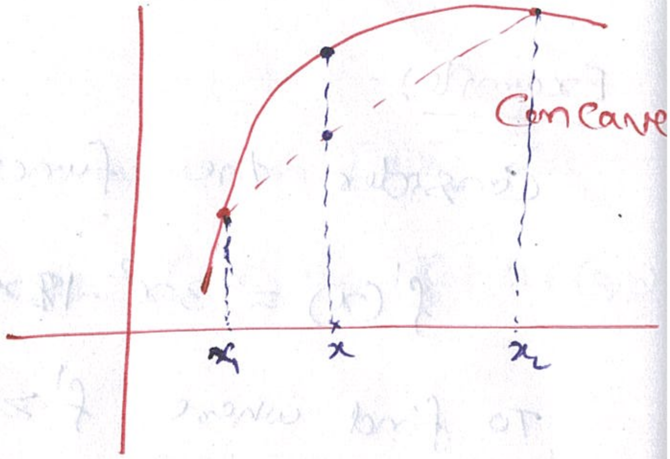



Defn- (Convex)

A fn. $f: I \to \mathbb{R}$ is said to be convex (or concave up) on $I$ if for any $x_1 < x < x_2$ in $I$,

$$f(x) \leq f(x_1) + \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}(x - x_1)$$




Here $m = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

**Def:- (Concave)**

A fn. $f: I \to \mathbb{R}$ is said to be Concave on $I$ if for any $x_1 < x < x_2$ on $I$, we have

$$f(x) \geq f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1).$$

**Property:-**

1. Let $I$ be an interval & $f: I \to \mathbb{R}$ be a differentiable function. Then

   i) $f'$ is increasing on $I \Leftrightarrow f$ is Convex on $I$

   ii) $f'$ is decreasing on $I \Leftrightarrow f$ is Concave on $I$

2. Let $I$ be an interval & $f: I \to \mathbb{R}$ be twice differentiable function. Then

   i) $f'' \geq 0$ on $I \Leftrightarrow f$ is convex on $I$

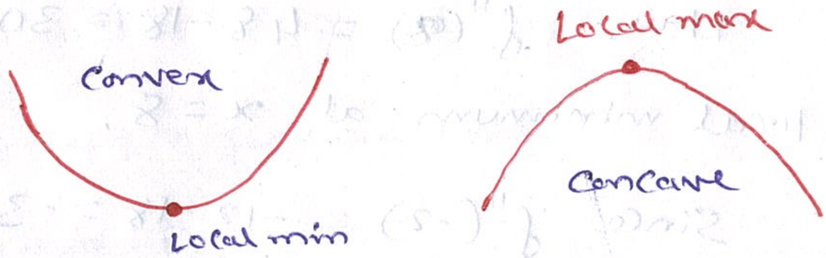
   ii) $f'' \leq 0$ on $I \Leftrightarrow f$ is concave on $I$.

Note:-

From the above two properties, we can observe that if $p$ is a critical point of $f$, with $f'(p)=0$, then the graph of $f$ has a horizontal tangent line at $p$.

Hence i, If the graph is convex at $p$, then $f$ has a local minimum at $p$.

ii, If the graph is concave at $p$, then $f$ has a local maximum.




Using the above properties and the note, we have the second-derivative test for local maxima and minima :-

i, If $f'(p)=0$ and $f''(p)>0$ then $f$ has a local minimum at $p$.

ii, If $f'(p)=0$ and $f''(p)<0$ then $f$ has a local maximum at $p$.

iii, If $f'(p)=0$ and $f''(p)=0$ then the test tells us nothing.

Example:-

1. $f(x) = x^3 - 9x^2 - 48x + 52$.

$$f'(x) = 3x^2 - 18x - 48$$

and the critical points of $f$ are $x = -2$ and $x = 8$ ($\because f'(x) = 0 \Rightarrow x = -2$ & $x = 8$)

We have $f''(x) = 6x - 18$

Thus $f''(8) = 48 - 18 = 30 > 0$, so $f$ has a local minimum at $x = 8$.

Since $f''(-2) = -12 - 18 = -30 < 0$, $f$ has a local maximum at $x = -2$.

Note:- The second derivative test does not tell us anything if both $f'(p) = 0$ and $f''(p) = 0$.

Ex:- If $f(x) = x^3$ and $g(x) = x^4$, both

$$f'(0) = g'(0) = 0 \text{ and } f''(0) = g''(0) = 0.$$

Here the pt. $x = 0$ is a minimum for $g$ but is neither minimum nor a maximum for $f$.

However, the first derivative test is still useful. For example, $g'$ changes sign from negative to positive at $x = 0$, so we know $g$ has a local minimum there.

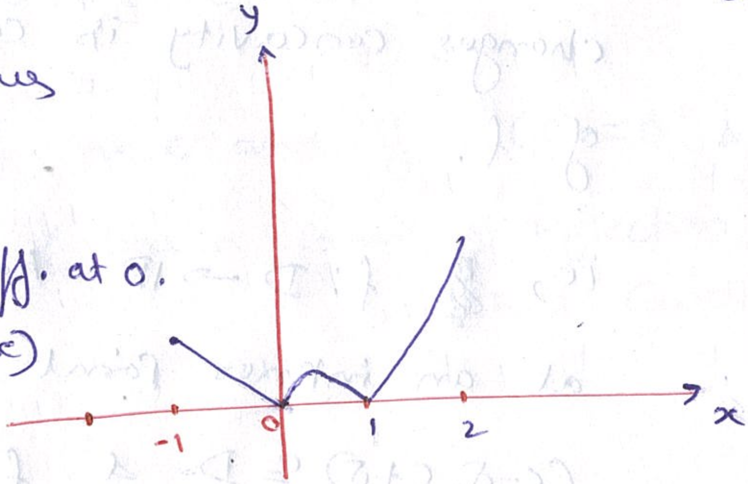
Example:-

1. Define $f : [-1, 2] \to \mathbb{R}$ by $f(x) = \begin{cases} -x & \text{if } -1 \le x \le 0 \\ 2x^3 - 4x^2 + 2x & \text{if } 0 < x \le 2 \end{cases}$

Clearly $f$ is continuous on $[-1, 2]$.

However, $f$ is not diff. at 0.

(Take this as an excercise)




Also,

$$f'(x) = -1 \quad \text{for } -1 \le x < 0 \text{ \&}$$

$$f'(x) = 6x^2 - 8x + 2 \quad \text{for } 0 < x \le 2$$

$$= 2(3x-1)(x-1).$$

Now $f'(x) = 0$ when $x = 0, \frac{1}{3}, 1$.

$\therefore$ The critical points are $0, \frac{1}{3}, 1$

& End points $-1, 2$.

We can find the absolute max, absolute minimum ~~at~~ by checking the ~~following~~ value of $f$ at the following pts.

$-1, 2, 0, \frac{1}{3}, 1$.

Hence

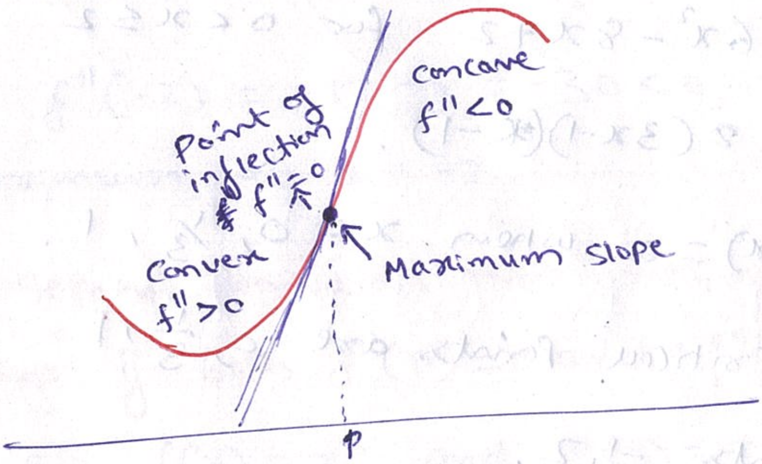
| $x$ | $-1$ | $0$ | $\frac{1}{3}$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $1$ | $0$ | $8/27$ | $0$ | $4$ |

$\therefore$ $f$ has absolute max. at $x = 2$ & absolute min. at $x = 0$ & $1$.

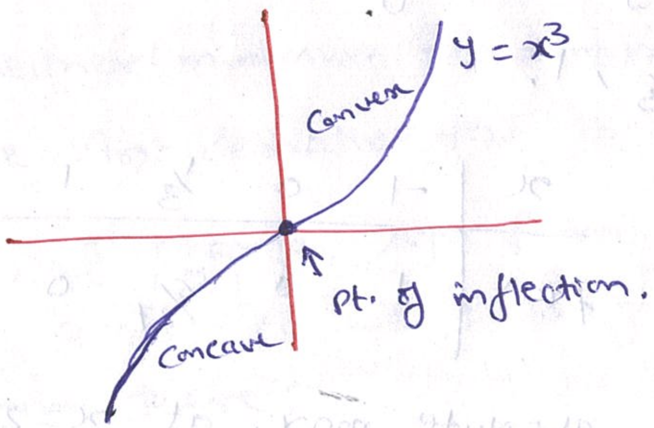
# Inflection Points:

**Def:-** A point at which the graph of a function changes concavity is called an inflection point of f.

ie, $f: D \to \mathbb{R}$ has an inflection point at an interior point $c$ of $D$ if $\exists\ \delta > 0 \ni (c-\delta, c+\delta) \subseteq D$ & $f$ is convex on $(c-\delta, c)$ and concave on $(c, c+\delta)$ or vice-versa.




**Ex:-** $f(x) = x^3$ has a point of inflection at 0.



Note:-

1. If $f$ is twice diff. at $c$ and $f$ has a pt. of inflection at $c$, then $f''(c) = 0$. [Necessary Condition]

2. If $f$ is twice diff. at $c$ and if $f''(c) = 0$ & $f'''(c) \neq 0$, then $f$ has a point of inflection at $c$. [Sufficient Condition].

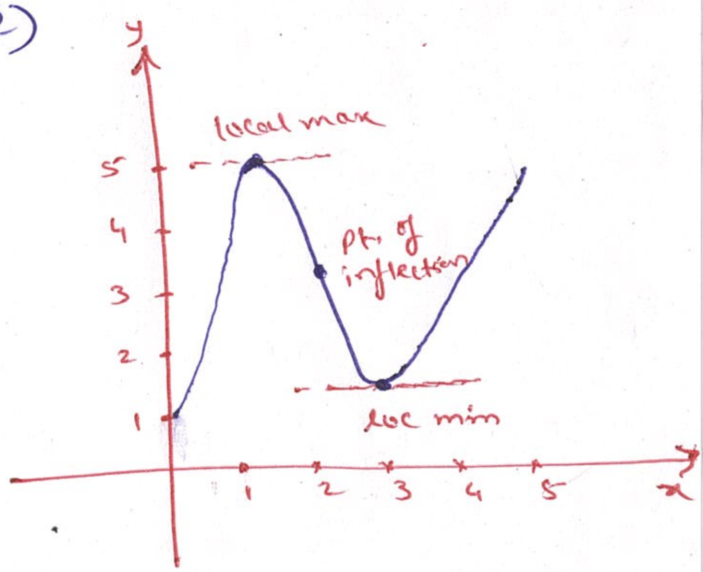
Example:-

$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 6x^2 + 9x + 1$.

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

$f''(x) = 6x - 12 = 6(x-2)$

| | $(-\infty, 1)$ | $(1, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| $f'$ | +ve | -ve | +ve |
| $f$ | increasing | decreasing | increasing. |




| | $(-\infty, 2)$ | $(2, \infty)$ |
|---|---|---|
| $f''$ | -ve | +ve |
| $f$ | Concave | Convex |